# UNIT III- UNIFORMLY ACCELERATING PARTICLE MODEL

## INSTRUCTIONAL GOALS

1. **Concepts of acceleration, average vs instantaneous velocity**
   Contrast graphs of objects undergoing constant velocity and constant acceleration
   Define instantaneous velocity (slope of tangent to curve in **x** vs **t** graph)
   Distinguish between instantaneous and average velocity
   Define acceleration, including its vector nature
   Motion map now includes acceleration vectors

2. **Multiple representations (graphical, algebraic, diagrammatic)**
   Introduce stack of kinematic curves
   - position vs. time (slope of tangent = instantaneous velocity)
   - velocity vs. time (slope = acceleration, area under curve = change in position)
   - acceleration vs. time (area under curve = change in velocity)

   Relate various expressions

3. **Uniformly Accelerating Particle model**
   Domain and kinematical properties
   Derive the following relationships from **x** vs t and **v** vs t graphs

| | | |
|---|---|---|
| $\vec{a} \equiv \dfrac{\Delta \vec{v}}{\Delta t}$ | Eq. 1 | definition of average acceleration |
| $\vec{v} = \vec{v}_0 + \vec{a}t$ | Eq. 2 | linear equation for a **v**-t graph |
| $\vec{v}_f = \vec{v}_i + \vec{a}\Delta t$ | Eq. 3 | generalized equation for any $t_i$ to $t_f$ interval |
| $\vec{x} = \vec{x}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$ | Eq. 4 | parabolic equation for an **x**-t graph |
| $\vec{x}_f = \vec{x}_i + \vec{v}_i \Delta t + \frac{1}{2}\vec{a}\Delta t^2$ | Eq. 5 | generalized equation for any $t_i$ to $t_f$ interval |
| $\vec{v}_f^2 = \vec{v}_i^2 + 2\vec{a}\Delta\vec{x}$ | Eq. 6 | algebraic combination of equations 3 and 5 |

4. **Analysis of free fall**

5. **Software**
   *Conceptual Kinemtaics Tutorial*
   *Graphs and Tracks* (Physics Academic Software)

# SCOPE AND SEQUENCE

1. Lab: Inclined Rail Motion
   Develop the position proportional to time-squared relation during whiteboarding.
   Post-lab: Use tangents to the x-t graph curve to create a v-t graph and define instantaneous velocity.

2. Lab Extension
   Analyze area under v-t graph to develop remaining mathematical kinematic expressions.

3. Optional Lab Extension: Uniformly accelerated motion worksheet
   Relate the average velocity of a time interval to the instantaneous velocity at the middle time.
   **(Your decision to use this will depend on your students and the amount of time you wish to spend on this unit.)**

4. Lab Deployment: Speeding up and slowing down
   Practice connections between x-t, v-t and a-t graphs.
   Sign of acceleration depends on speeding up/slowing down and sign of displacement

5. Computer tutorial: Graphs and Tracks II, Graphs and Tracks I
   Practice connections between x-t, v-t and a-t graphs.
   (The Conceptual Kinematics Tutorials can be interspersed at appropriate points throughout the unit.)

6. Worksheet 1

7. Worksheet 2

8. Worksheet 3

9. Lab: Free fall motion

10. Worksheet: Acceleration due to gravity on planet Newtonia

11. Worksheet 4

12. Worksheet/review sheet: Stacks of kinematic curves

13. Uniformly Accelerating Particle Test

# Lab Notes: Inclined Rail Motion

## Apparatus

**High Tech.**
Pasco dynamics carts and tracks or large steel ball
photogates (2)
Computer and ULI interface
ULI Timer (software for Macintosh)
Graphical Analysis

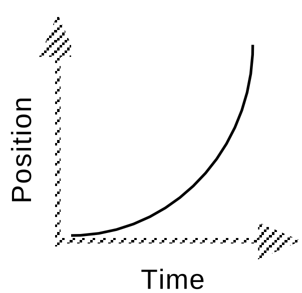
**Low Tech.**
Bowling ball, chalk and accessibility ramp or
disc and axle and parallel-pipe ramp
Stopwatch, water clock, metronome or pendulum
Ticker tape, masking tape and markers
Graphical Analysis

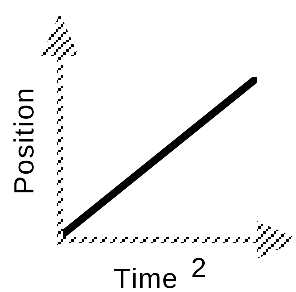
## Pre-lab discussion

- Let a ball roll down an inclined rail and ask students for observations. Record all observations. To proceed, they must mention something to the effect that the ball speeds up as it rolls down.
- To obtain a finer description, ask students which observations are measurable. Make sure they include the observation that the ball speeds up as it rolls down the rail. (Do not let them state the ball accelerates since we haven't defined acceleration yet!)
- Ask them how they can measure speed directly. Lead them to the conclusion that they cannot, but that they can measure position and time.
- Students should mark the position of the object at equal time intervals.
- Time should be plotted as the independent variable.

## Lab performance notes

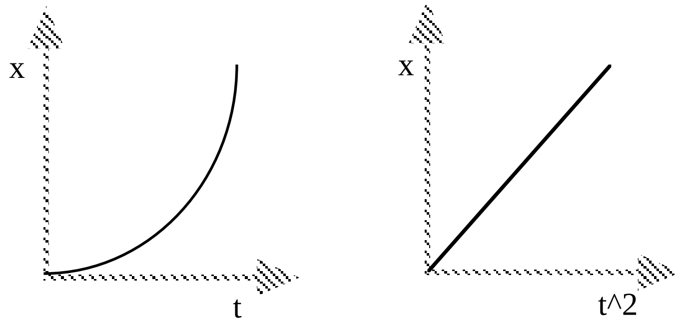
- A variety of constant acceleration motion such as a cart rolling down a track, a bowling ball rolling down an access ramp, or a disc and axle rolling down a ramp of two parallel pieces of conduit pipe.
- Timing variations could include using photogates, water clocks, pendulums and metronomes in addition to stopwatches.
- Make sure that the angle of inclination is less than 30$^{o}$.
- Initial position and speed must be zero. (See sample graphs below.)



## Post-lab discussion

From the lab, the students have the following graphs.



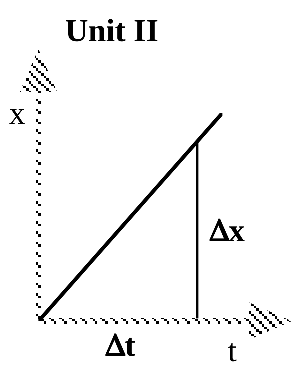

Focus the whiteboard discussion on their experimental procedure and the verbal interpretation of the parabolic x-t graph. Students should be able to describe that the displacement during each time interval increases over the previous time interval. Since the object travels greater distances in each successive time interval, the velocity is increasing.

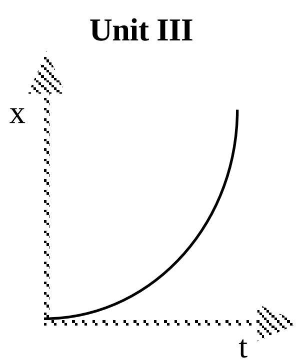
They should have also written an expression for the straight-line graph: $x = kt^2 + b$, where $b \to 0$. The units of the constant of proportionality (slope) are $m/s^2$, but k is **not** the acceleration of the object. Emphasize the x vs $t^2$ relationship and correct use of units, but stating that the slope has the units of acceleration would be premature, because that quantity has yet to be defined.

# Post-lab extension

## Discussion

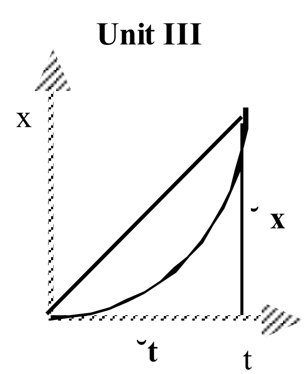
Contrast the **x** vs **t** graph for this lab with the one obtained in unit II.



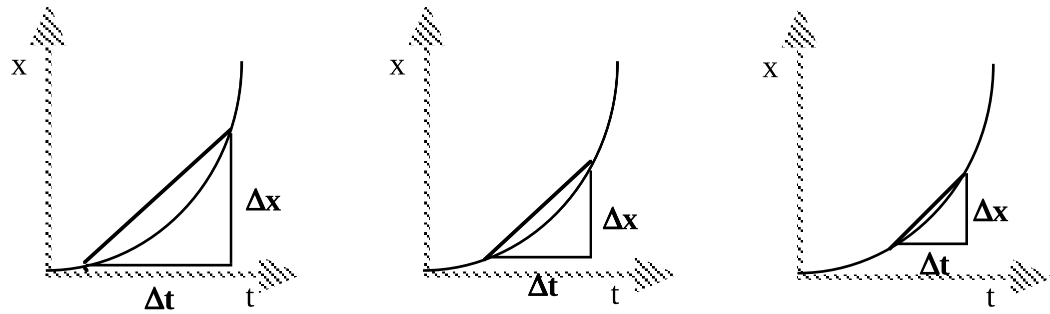




One can speak of the average velocity as the slope of the graph (above left) because the slope of a straight line is constant. It doesn't matter which two points are used to determine the slope.

On the other hand, one could speak of the average velocity of the object in the graph at right, but since the object started very slowly and steadily increased its speed, the term average velocity has little meaning.

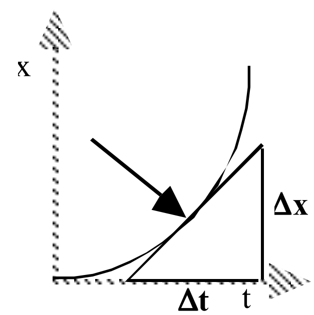


What would be more useful is to have a way of describing the object's speed at a given instant (or as Arons terms it: *clock reading*). To develop this idea, you must show that, as you shrink the time interval Δt over which you calculate the average velocity, the secant (line intersecting the curve at two points) more closely resembles the curve during that interval.




That is, the slope of the secant gives the average velocity for that interval. As the interval gets shorter and shorter, the secant more closely approximates the curve. Thus, the average velocity of this interval becomes a more and more reasonable estimate of how fast the object is moving *at any instant* during this interval.

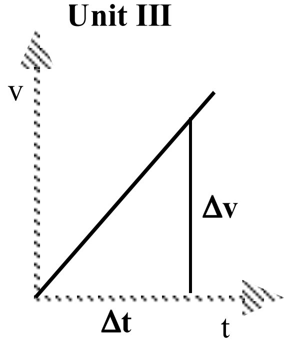
As one shrinks the interval, Δt to zero, the secant becomes a tangent; the slope of the tangent is the average velocity at this instant, or simply the *instantaneous velocity* at that clock reading.



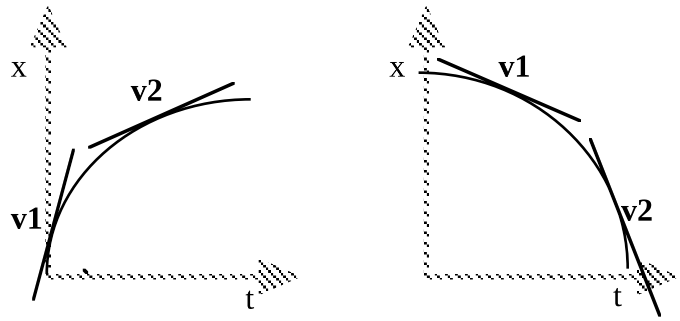

## Student activity

Using the x-t graphs the students produced in the lab, students should construct at least five tangents to the curve and determine the slope of each tangent. This will be easier if the students have printed out full-page graphs of their data or replotted their data on a sheet of graph paper. The students should then make a new graph of instantaneous velocity vs time. An alternative is to have students enter their equations into a graphing calculator and have the calculator draw tangents and determine their slopes at five clock readings. (See TI-slope.doc or CASIO-slope.doc)

**Unit III**



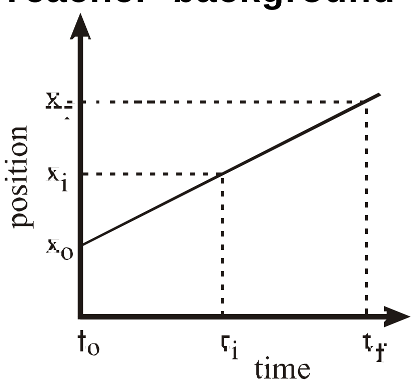

A plot of instantaneous velocity (v, instead of v-bar) vs time should yield a straight line. The slope of this line is $\frac{\Delta \vec{v}}{\Delta t} \equiv \bar{a}$. That is, the change in velocity during a given time interval is defined to be the *average acceleration*. The equation for the line can be written as $\vec{v} = \vec{a}t + \vec{v}_0$, where $\vec{v}_0$ is the y-intercept. It is important to define the acceleration this way, and then show examples of **x** vs **t** graphs in which the acceleration is negative.




In both cases, $v_2 - v_1$ is negative, yet very different situations are being represented. We advise against the use of the term *deceleration*, because students invariably think that this term implies negative acceleration ***means*** slowing down; the two conditions are not synonymous.

Generalizing the linear v-t equation for any time interval $t_i$ to $t_f$ yields $\vec{v}_f = \vec{v}_i + \vec{a}\Delta t$ . The development of this expression is provided below to clarify the use of $\Delta t$ as opposed to t.

## Teacher background




The slope is defined to be average velocity.

$$\frac{\Delta \vec{x}}{\Delta t} \equiv \vec{\bar{v}} \qquad \textit{Eq.1}$$

Equation of the line

$$\vec{x} = \vec{v}t + \vec{x}_0 \qquad \textit{Eq.2}$$

Generalize the equation for the interval $t_i$ to $t_f$.

At $t_f$:

$$\vec{x}_f = \vec{v}t_f + \vec{x}_0 \qquad \textit{Eq.3}$$

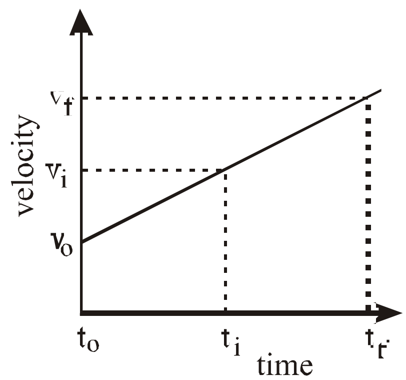
At $t_i$:

$$\vec{x}_i = \vec{v}t_i + \vec{x}_0 \qquad \textit{Eq.4}$$

Subtract equation 4 from 3:

$$\vec{x}_f - \vec{x}_i = \vec{v}(t_f - t_i) + \vec{x}_0 - \vec{x}_0$$

$$\vec{x}_f = \vec{x}_i + \vec{v}\Delta t \qquad \textit{Eq.5}$$




The slope is defined to be average acceleration.

$$\frac{\Delta \vec{v}}{\Delta t} \equiv \vec{\bar{a}} \qquad \textit{Eq.6}$$

Equation of the line

$$\vec{v} = \vec{a}t + \vec{v}_0 \qquad \textit{Eq.7}$$

Generalize the equation for the interval $t_i$ to $t_f$.

At $t_f$:

$$\vec{v}_f = \vec{a}t_f + \vec{v}_0 \qquad \textit{Eq.8}$$

At $t_i$:

$$\vec{v}_i = \vec{a}t_i + \vec{v}_0 \qquad \textit{Eq.9}$$

Subtract equation 9 from 8:

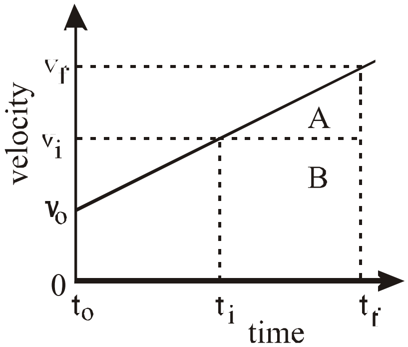
$$\vec{v}_f - \vec{v}_i = \vec{a}(t_f - t_i) + \vec{v}_0 - \vec{v}_0$$

$$\vec{v}_f = \vec{v}_i + \vec{a}\Delta t \qquad \textit{Eq.10}$$

# Post-lab Extension (Development of Kinematic Expressions)

Developing the remaining kinematic equations involves finding the area under a v-t graph and algebraic combination of equations. Depending on the ability of your students, various levels of guidance can be provided to help the students derive the equations themselves. Don t get hung up on the algebra. Focus on the physics.

The displacement of a uniformly accelerating object is equivalent to the area under the v-t graph. In this situation, we are interested in the displacement during the time interval $t_i$ to $t_f$.




<u>Area of region A:</u>

1/2 *height x base* area of a triangle

$\frac{1}{2} \cdot (\vec{v}_f - \vec{v}_i) \cdot (t_f - t_i)$

$\frac{1}{2} \cdot \Delta\vec{v} \cdot \Delta t$

$\frac{1}{2} \cdot \vec{a}\Delta t \cdot \Delta t$ substitute $\vec{v} = \vec{a}\Delta t$

$\frac{1}{2}\vec{a}(\Delta t)^2$

<u>Area of region B</u>

*length x width* area of a rectangle

The velocity at the horizontal axis is zero;

$(\vec{v}_i - 0) \cdot (t_f - t_i) = \vec{v}_i \Delta t$

<u>The total displacement</u> is equal to A + B.

$\Delta\vec{x} = \frac{1}{2}\vec{a}\Delta t^2 + \vec{v}_i\Delta t$

Rearranging:

$\vec{x}_f = \vec{x}_i + \vec{v}_i\Delta t + \frac{1}{2}\vec{a}\Delta t^2$ *Eq.11*

Combining equations 6 and 11 produces a time-independent kinematics expression.

$$\vec{a} \equiv \frac{\Delta\vec{v}}{\Delta t} \quad \textit{Eq. 6}$$

Rearrange:

$$\Delta t = \frac{\Delta\vec{v}}{\vec{a}} \;;\; \Delta t = \frac{\vec{v}_f - \vec{v}_i}{\vec{a}} \quad \textit{Eq. 12}$$

$$\vec{x}_f = \vec{x}_i + \vec{v}_i\Delta t + \tfrac{1}{2}\vec{a}\Delta t^2 \quad \textit{Eq. 11}$$

$$\Delta\vec{x} = \vec{v}_i\Delta t + \tfrac{1}{2}\vec{a}\Delta t^2$$

Substitute equation 12 into equation 11:

$$\Delta\vec{x} = \vec{v}_i\left[\frac{\vec{v}_f - \vec{v}_i}{\vec{a}}\right] + \tfrac{1}{2}\vec{a}\left[\frac{\vec{v}_f - \vec{v}_i}{\vec{a}}\right]^2$$

Multiply both sides by $2\vec{a}$

$$2\vec{a}\Delta\vec{x} = 2\vec{v}_i(\vec{v}_f - \vec{v}_i) + (\vec{v}_f - \vec{v}_i)^2$$

Multiply out the terms on the right.

$$2\vec{a}\Delta\vec{x} = 2\vec{v}_i\vec{v}_f - 2\vec{v}_i^2 + \vec{v}_f^2 - 2\vec{v}_i\vec{v}_f + \vec{v}_i^2$$

Simplify the right side of equation

$$2\vec{a}\Delta\vec{x} = -\vec{v}_i^2 + \vec{v}_f^2$$

Rearrange:

$$\vec{v}_f^2 = \vec{v}_i^2 + 2\vec{a}\Delta\vec{x} \quad \textit{Eq. 13}$$

Summary of mathematical models:

| | | |
|---|---|---|
| $\vec{a} \equiv \frac{\Delta \vec{v}}{\Delta t}$ | Eq. 6 | definition of average acceleration |
| $\vec{v} = \vec{v}_0 + \vec{a}t$ | Eq. 7 | linear equation for a **v**-t graph |
| $\vec{v}_f = \vec{v}_i + \vec{a}\Delta t$ | Eq. 10 | generalized equation for any $t_i$ to $t_f$ interval |
| $\vec{x} = \vec{x}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$ | | parabolic equation for an **x**-t graph |
| $\vec{x}_f = \vec{x}_i + \vec{v}_i\Delta t + \frac{1}{2}\vec{a}\Delta t^2$ | Eq. 11 | generalized equation for any $t_i$ to $t_f$ interval |
| $\vec{v}_f^2 = \vec{v}_i^2 + 2\vec{a}\Delta\vec{x}$ | Eq. 13 | algebraic combination of equations 3 and 5 |

## Optional lab extension: Uniformly accelerated motion worksheet

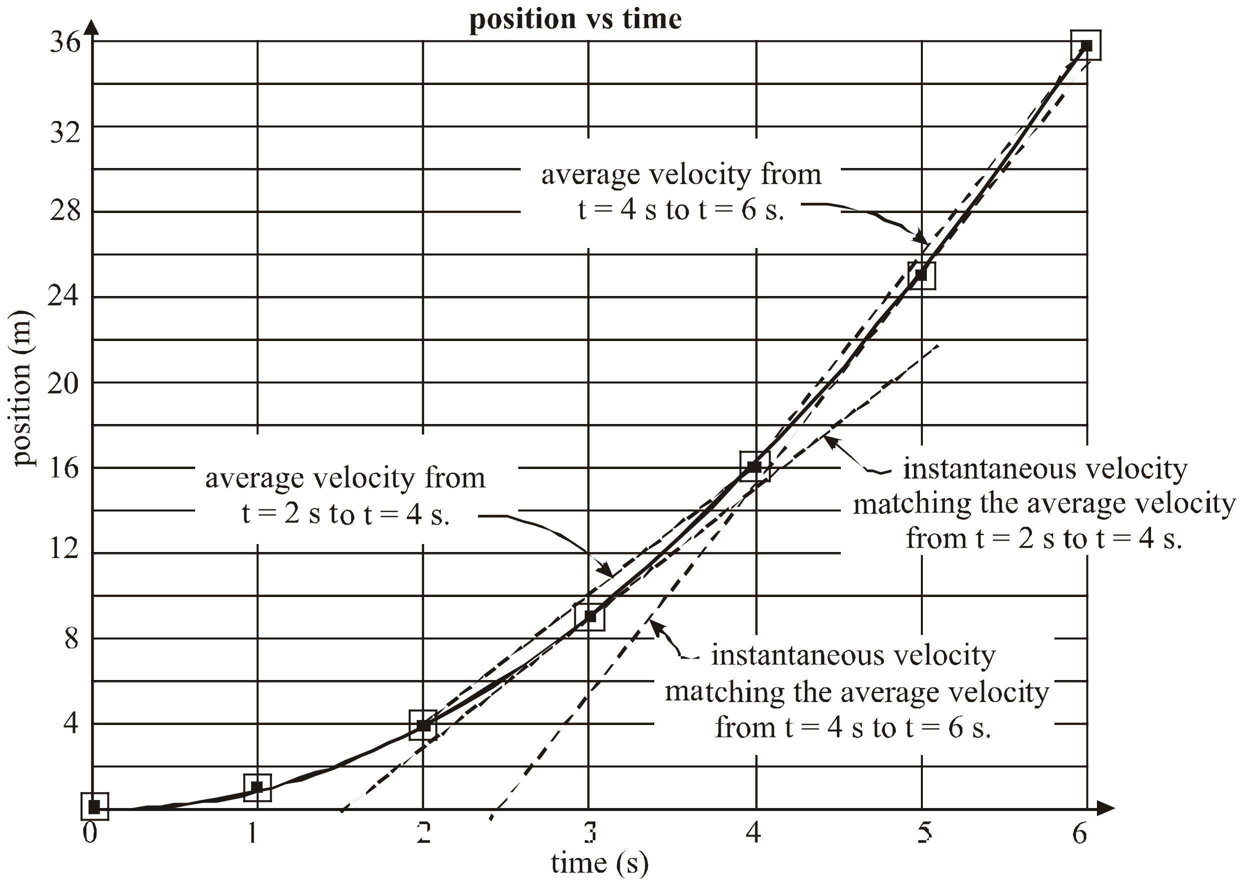
**(your decision to use this will depend on your students and the amount of time you wish to spend on this unit).**

Students will see that the creating v-t graphs by finding the slope of a sequence of tangents to the x-t graph is a bit tedious. However, the instantaneous velocity can also be found mathematically from a sequence of position-time data. The desired goal of the activity is to establish that for uniformly accelerated motion, *the average velocity for a time interval is equal to the instantaneous velocity at the clock reading in the middle of the time interval.* This idea is important since it is the same algorithm used by computer motion analysis programs such as Mac Motion, Logger Pro, and Science Workshop. Students can use the reasoning developed here to analyze the motion of a picket fence falling through a photogate.

Use a thought experiment with small numbers (an acceleration of 2 $m/s^2$) to build the first two columns of simulated x-t data, leaving a space between each row. Utilize the expression $\vec{v}_f = \vec{v}_i + \vec{a}\Delta t$ to lead the students to identifying the instaneous velocity at each time. Since the initial velocity is zero, the average velocity from t = 0 to t = $t_f$ is half the instantaneous velocity at t = $t_f$ and the displacement is the average velocity times Δt. For example, at t = 3 s the instaneous velocity is 2 $m/s^2$ x 3 s or 6 m/s. The average velocity from t = 0 to t = 3 s is 3 m/s and the displacement is 3 m/s x 3 s or 9 m.

| t (s) | x (m) | Δt (s) | Δx (m) | $v_{average}$ (m/s) | $t_{middle\ of\ interval}$ (s) |
|---|---|---|---|---|---|
| 0.0 | 0.0 | | | | |
| | | | | | |
| 1.0 | 1.0 | | | | |
| | | | | | |
| 2.0 | 4.0 | | | | |
| | | | | | |
| 3.0 | 9.0 | | | | |
| | | | | | |
| 4.0 | 16 | | | | |
| | | | | | |
| 5.0 | 25 | | | | |
| | | | | | |
| 6.0 | 36 | | | | |

After completing the first two columns of data, analyze the resulting x-t graph. The goal is to find the correlation between average velocity during a time interval and instantaneous velocity at a clock reading.




The average velocity from t = 2 s to t = 4 s:

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{16m - 4m}{4s - 2s} = 6\,\tfrac{m}{s}$$

From the graph, it appears that the slope of the chord connecting t = 2 s to t = 4 s has the same slope as the tangent to the curve at t = 3 s. The slope of the tangent at t = 3 s:

$$v = \frac{\Delta x}{\Delta t} = \frac{21m - 0m}{5s - 1.5s} = 6\,\tfrac{m}{s}$$

Let s try the comparison of average and instantaneous velocities on a different part of the graph.

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{36m - 16m}{6s - 4s} = 10\,\tfrac{m}{s}$$

From the graph, it appears that the slope of the chord connecting t = 4 s to t = 6 s has the same slope as the tangent to the curve at t = 5 s. The slope of the tangent at t = 5 s:

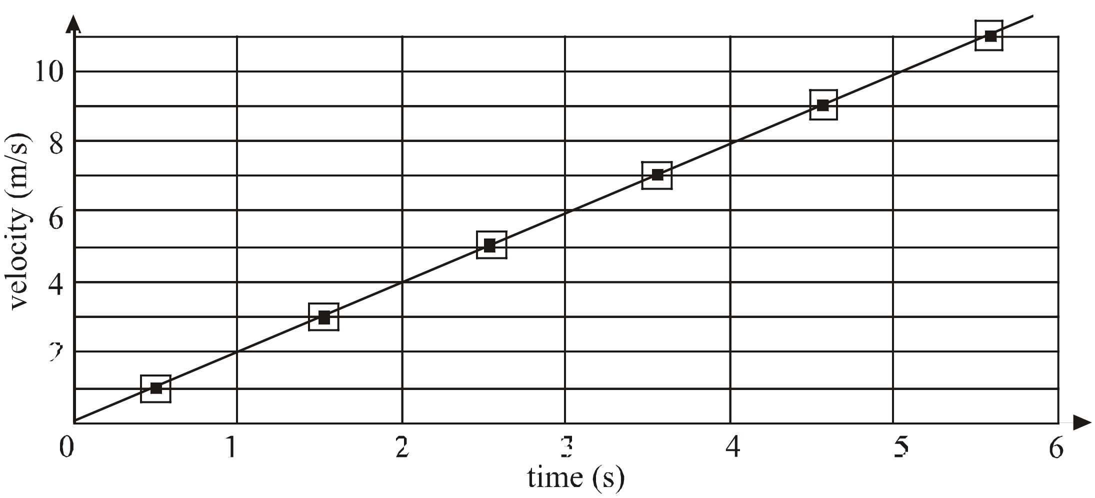
$$v = \frac{\Delta x}{\Delta t} = \frac{35m - 0m}{6s - 2.5s} = 10\,\tfrac{m}{s}$$

Although the examples above do not constitute a rigorous proof, they provide a strong argument for the following conclusion: *for uniformly accelerated motion, the average velocity for a time interval is equal to the instantaneous velocity at the clock reading in the middle of the time interval.*

Calculations can now be made to fill in the rest of table below:

| t (s) | x (m) | Δt (s) | Δx (m) | $v_{average}$ (m/s) | $t_{middle\ of\ interval}$ (s) |
|---|---|---|---|---|---|
| 0.0 | 0.0 | | | | |
| | | 1.0 | 1.0 | 1 | 0.5 |
| 1.0 | 1.0 | | | | |
| | | 1.0 | 3.0 | 3 | 1.5 |
| 2.0 | 4.0 | | | | |
| | | 1.0 | 5.0 | 5 | 2.5 |
| 3.0 | 9.0 | | | | |
| | | 1.0 | 7.0 | 7 | 3.5 |
| 4.0 | 16 | | | | |
| | | 1.0 | 9.0 | 9 | 4.5 |
| 5.0 | 25 | | | | |
| | | 1.0 | 11.0 | 11 | 5.5 |
| 6.0 | 36 | | | | |

Using the idea that the average velocity during a time interval is equal to the instantaneous velocity at the clock reading in the middle of the time interval, an instantaneous velocity vs time graph can produced.




The students now have a mathematical way of producing a v-t graph from their measurements of position and time. Analyzing the v-t graph allows determination of acceleration and displacement with or without an initial velocity. With this background, students can begin the uniformly accelerated motion worksheet.

# Deployment lab: (Speeding Up and Slowing Down)

## Apparatus

Motion detector-interface-computer
Ramp and oatmeal can or #10 tin can *or*
Pasco cart and track *or*
Pasco fan cart and track.

## Pre-lab discussion

Make sure students understand the format of the lab. They are to observe the motion and then draw a motion map and predicted graphs. Then the students check their graph predictions utilizing the motion detector.

It may be helpful to show students how to resize the graph axes so students can focus on the relavant portions of the graph. Analysis should focus on the region of the graph in which the cart is coasting and not on initial pushes or final stops.

## Lab-performance notes

Perform the lab ahead of time and create a template file the students can use. The template should display x-t, v-t and a-t graphs each with appropriately scaled axes for your situation.

Situation 6 requires a separate template where the zero position is in the middle of the track. This forces students to confront the fact that the sign of the <u>change</u> in position, not position itself, determines the sign of the velocity.

## Post-lab discussion

Use the whiteboard session to reinforce connections between the actual motion, the description of the motion, the motion map and the x-t, v-t and a-t graphs.

It may be helpful to refer to a pencil as a  slope indicator  and have students hold the center of the pencil tangent to various places on the curve. The slope of the pencil is the instantaneous velocity on the x-t graph, and and the instantaneous acceleration on a v-t graph. Moving the pencil along the curve on the graph and observing how the slope of the tangent changes can help students to see how the velocity or acceleration changes.

Questions on the relations between graphs can be based on the following summary from the PSSC text:

x-t —slope→ v-t —slope→ a-t

a-t —area→ Δ v-t —area→ Δ x-t

a-t —area and initial v→ v-t —area and initial x→ x-t

# Deployment activity: free fall and picket fence

## Apparatus

Large picket fence
Photogate
Lab timing software
(Logger-Pro, ULI Timer, Science Workshop)

Alternatives:
Strobe photo
Tossing a basketball above a motion detector
Video camera analysis

### Pre-lab discussion

¥ Review the ball/rail or car/track lab. In each case, the acceleration depended on the incline. Ask how to maximize acceleration.

¥ Demonstrate how picket fence moving through photogate yields $\Delta x$ vs $\Delta t$ data; $\Delta x$ is constant (0.05m), but, if object is accelerating, $\Delta t$ decreases.

¥ Introduce feature of Timing software that will produce **v** vs **t** graph. Induce students to offer that slope of this graph equal the acceleration of a freely falling object.

### Lab performance notes

¥ Be sure that the picket fence lands on a soft surface.

### Post-lab discussion

¥ Have students print out **v** vs **t** graph. Compare value of acceleration with those from other groups.

¥ Induce them to provide an explanation for the existence of a y-intercept. What is its physical significance?

¥ You may want to show the second part of the Mechanical Universe video, The Law of Falling Bodies.